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ECS315 2019/1 Part III.1 Dr.Prapun

7 Random variables

In performing a chance experiment, one is often not interested in the particular outcome that occurs but in a specific numerical value associated with that outcome. In fact, for most applications, measurements and observations are expressed as numerical quantities.

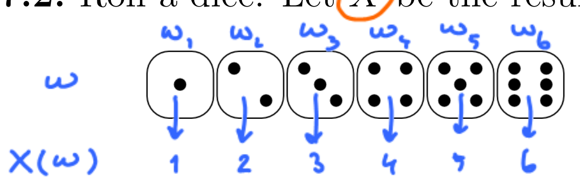
Example 7.1. Take this course and observe your grades.

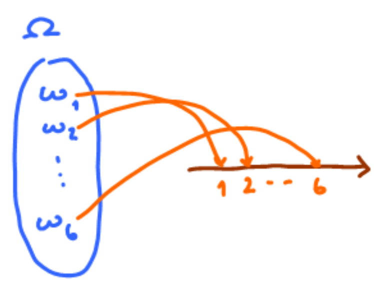
$$\Omega = \{A, B+, B, C+, C, D+, D, F\}$$

Define a function $G(\cdot)$ that maps the letter grades to numerical values:

$$G(A) = 4, G(B+) = 3.5, G(B) = 3, G(C+) = 2.5, G(C) = 2,$$
$$G(D+) = 1.5, G(D) = 1, G(F) = 0.$$

Example 7.2. Roll a dice. Let X be the result.





7.3. The advantage of working with numerical quantities is that we can perform mathematical operations on them.

→ add, subtract, multiply, divide, average, max, min

In the mathematics of probability, averages are called expectations or expected values.

7.4. *Intuitively*, a random variable is a "variable" that "takes on its values by chance".

7.5. The convention is to use capital letters such as X, Y, Z to denote random variables.

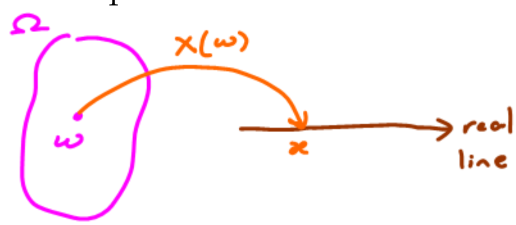
Definition 7.6. A real-valued function $X(\omega)$ defined for all points ω in a sample space Ω is called a ***random variable*** (RV) [29].

- A random variable is a rule that assigns a numerical value to each possible outcome of a chance experiment.

$f : D \to R$

$X : \Omega \to \mathbb{R}$

sample space



$y = f(x)$

$x = X(\omega)$

Example 7.7. Roll a fair dice:

$$\Omega = \{1, 2, 3, 4, 5, 6\} = \{\omega_i : \quad \omega_i = i, \quad i = 1, 2, \ldots, 6\}.$$

$X(\omega) = \omega$

$Y(\omega) = (\omega - 3)^2$

$Z(\omega) = \sqrt{Y(\omega)}$

$U(\omega) = \begin{cases} 1, & \omega \geq 4, \\ 0, & \omega < 4. \end{cases}$

Observation:

(a) More than one random variables can be defined on one sample space.

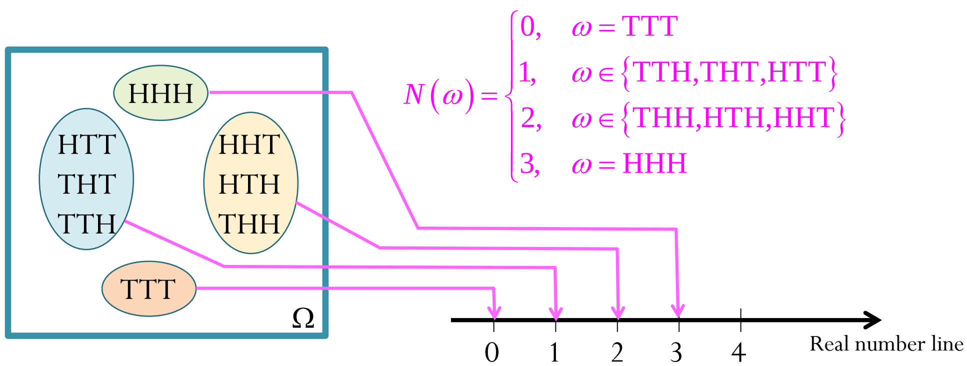
(b) Although the function X, Y, Z, and U are deterministically defined, their values depend on the value of the outcome from the experiment, which is random.

[29] The term "random variable" is a misnomer. Technically, if you look at the definition carefully, a random variable is a deterministic function; that is, it is not random and it is not a variable. [Toby Berger][25, p 254]

- As a function, it is simply a rule that maps points/outcomes ω in Ω to real numbers.
- It is also a deterministic function; nothing is random about the mapping/assignment. The randomness in the observed values is due to the underlying randomness of the argument of the function X, namely the experiment outcomes ω.
- In other words, the randomness in the observed value of X is induced by the underlying random experiment, and hence we should be able to compute the probabilities of the observed values in terms of the probabilities of the underlying outcomes.

Example 7.8 (Three Coin Tosses)**.** Counting the number of heads in a sequence of three coin tosses.

$$\Omega = \{\text{TTT},\text{TTH},\text{THT},\text{THH},\text{HTT},\text{HTH},\text{HHT},\text{HHH}\}$$



Example 7.9 (Sum of Two Dice)**.** If S is the sum of the dots when rolling one fair dice twice, the random variable S assigns the numerical value $i+j$ to the outcome (i, j) of the chance experiment.

Definition 7.10. Probability involving a random variable X will be expressed in the form

$$P\left[\text{some statement(s) about } X\right]. \tag{11}$$

Note the use of square brackets. Technically, when we write

$$\left[\text{some statement(s) about } X\right], \tag{12}$$

we are actually defining an event containing the outcomes ω that make $X(\omega)$ satisfy the given statement(s).

Now that we have an event, we can apply the steps in Chapter 5 to find its probability.

Additional details regarding this notation will be discussed in 7.15. We shall see later in 7.16 and 7.17 that any statement(s) about X can be expressed in the form $X \in B$. Hence, any event of the from 12 can be expressed concisely as $[X \in B]$ for some appropriate B.

Example 7.11. Continue from Example 7.7. Roll a fair dice. Let $\Omega = \{1, 2, 3, 4, 5, 6\}$. The dice is fair; therefore and the probability $P(\{\omega\}) = \frac{1}{6}$ for each outcome ω inside Ω.

(a) Define $X(\omega) = \omega$. Find $P[X = 4]$.

- Method 1: We are finding the probability that $X = 4$.
  The statement under consideration is "$X = 4$".
  From $X(\omega) = \omega$, $\quad X(\omega) = 4$ when $\omega = 4$.
  Therefore, $P[X = 4] = P(\{4\}) = \frac{1}{6}$.
- Method 2: By definition,
$$[X = 4] = \{\omega \in \Omega : X(\omega) = 4\}.$$
  Therefore,
$$P[X = 4] = P([X = 4]) = P(\{4\}) = \frac{1}{6}.$$

(b) Define $Y(\omega) = (\omega - 3)^2$. Find $P[Y = 4]$.

- Method 1: We are finding the probability that $Y = 4$.
  The statement under consideration is "$Y = 4$".
  From $Y(\omega) = (\omega - 3)^2$, $\quad Y(\omega) = 4$ when $\omega = 1$ or 5.
  Therefore, $P[Y = 4] = P(1) + P(5) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

$(\omega-3)^2 = 4$
$\omega - 3 = \pm 2$
$\omega = 3 \pm 2$
1 or 5

- Method 2: By definition,
$$[Y = 4] = \{\omega \in \Omega : Y(\omega) = 4\} = \left\{\omega : (\omega - 3)^2 = 4\right\}.$$
  Therefore,
$$P[Y = 4] = P([Y = 4]) = P(\{1, 5\})$$
$$= P(\{1\}) + P(\{5\}) = \frac{1}{3}.$$

Example 7.12. In Example 7.8 (Three Coin Tosses), if the coin is fair, then

$$P[N < 2] = P(\{TTT, TTH, THT, HTT\})$$
$$= 4 \times \frac{1}{8} = \frac{1}{2}$$

7.13. Summary: In Chapter 5, we studied how to find the probability of any event A by adding the probabilities of the individual outcomes inside A. For example,

$$P(\{a,b,c\}) = P(\{a\}) + P(\{b\}) + P(\{c\}).$$

In this chapter, we now have steps to find any probability involving a random variable when the random variable is explicitly defined as a function of outcomes.

Step 1: Identify the sample space Ω and the probability $P(\omega)$ for each outcome ω.

Step 2: Find the value of ω that makes $X(\omega)$ satisfy the given statement(s).

- For example, if we want to find the probability that $X = 3$, we need to find all $\omega \in \Omega$ that make $X(\omega) = 3$.
- The collection of such ω is the event that you are interested in.
- This basically turns the calculation into the one we know how to solve from Chapter 5.

Step 3: Find the probability of the event above.

- For example,

$$P(\{\omega_1, \omega_2, \omega_3\}) = P(\{\omega_1\}) + P(\{\omega_2\}) + P(\{\omega_3\}).$$

Example 7.14. [M2016Q11] Consider a sample space $\Omega = \{1, 3, 4\}$. Suppose, for $\omega = 1, 3, 4$, we have

$$P(\{\omega\}) = c\omega$$

ω	$P(\{\omega\}) = c\omega$
1	$c = 1/8$
3	$3c = 3/8$
4	$4c = 4/8 = 1/2$

for some constant c.

(a) Check that $c = 1/8$.

$P(\Omega) = 1 \Rightarrow P(\{1\}) + P(\{3\}) + P(\{4\}) = 1$

$c + 3c + 4c = 1 \Rightarrow c = \frac{1}{8}$

(b) Define $A = \{1, 3\}$ and $B = \{1, 4\}$.

(i) Find $P(A)$.

$P(A) = P(\{1\}) + P(\{3\}) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}$

(ii) Find $P(A \cap B)$. $= P(\{1\}) = \frac{1}{8}$

(c) Define a random variable X by $X(\omega) = \frac{12}{\omega}$.

(i) What are the possible values of X?

3, 4, 12

ω	$X(\omega)$
1	12
3	4
4	3

(ii) Find $P[X = 3]$.

$= P(\{4\}) = \frac{1}{2}$

(iii) Find $P[X > 3]$.

$= P(\{1, 3\}) = P(\{1\}) + P(\{3\}) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

Below, from 7.15 to 7.18, we will look deeper into the details of the concepts defined in Definition 7.10.

Definition 7.15. Events involving random variables: When we write

$$[\text{some statement(s) about } X],$$

we mean "the set of outcomes in Ω such that $X(\omega)$ satisfies the given statement(s)."

- $[X = x] = \{\omega \in \Omega : X(\omega) = x\}$
  - We usually use the corresponding lowercase letter[30] to denote
    (a) a possible value (realization) of the random variable
    (b) the value that the random variable takes on
    (c) the running values for the random variable
- $[X \in B] = \{\omega \in \Omega : X(\omega) \in B\}$
- $[a \le X < b] = \{\omega \in \Omega : a \le X(\omega) < b\}$
- $[X > a] = \{\omega \in \Omega : X(\omega) > a\}$

All of the above items are sets of outcomes. They are all events!

7.16. Event of the form "[some condition(s) on X]" or "[some statement(s) about X]" can be written in the from $[X \in B]$ for some appropriate B.

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[30] This is the same as writing $[X = c]$ where c is a constant. Basically, it is a generic notation for $[X = 5]$, $[X = 1.6]$, $[X = \pi]$, etc. We use this when

(a) we don't want to specify the constant in the expression yet or

(b) we want to say that the statement/equation/property containing it is valid for any value of c.

It turns out that, later on, we will have to deal with many random variables and hence it is convenient to have the name of the constant c match the name of the corresponding random variable. So, we talk about the events $[X = x]$, $[Y = y]$, and $[Z = z]$ instead of having to find new name for the constant corresponding to each one of them, say, $[X = c]$, $[Y = d]$, and $[Z = h]$.

You may think we can use constants $c_1, c_2, \ldots$. However, we also will have to deal with ransom variables $X_1, X_2, \ldots$, $Y_1, Y_2, \ldots$, $Z_1, Z_2, \ldots$. So, again, will have to come up with new names for a lot of constants.

Example 7.17. Expressing events in the form $[X \in B]$:

(a) $[5 \leq X < 8] = [X \in [5, 8)]$

(b) $[|X| < 3] = [X \in (-3, 3)]$

(c) $[X > 2] = [X \in (2, \infty)]$

(d) $[X = 1] = [X \in \{1\}]$

Definition 7.18. To avoid double use of brackets (round brackets over square brackets), we write $P[X \in B]$ when we means $P([X \in B])$. Hence,

$$P[X \in B] \equiv P([X \in B]) = P(\{\omega \in \Omega : X(\omega) \in B\}).$$

Similarly,

$$P[X < x] = P([X < x]) = P(\{\omega \in \Omega : X(\omega) < x\}).$$

Definition 7.19. A set S is called a ***support*** of a random variable X if $P[X \in S] = 1$.

- To emphasize that S is a support of a particular variable X, we denote a support of X by S_X.
- Practically, we usually define a support of a random variable X to be the set of all the "possible" values of X.[31]
- For any random variable, the set $\mathbb{R}$ of all real numbers is always its support; however, it is not quite useful because it does not further limit the possible values of the random variable.
- Recall that a support of a probability measure P is any set $A \subset \Omega$ such that $P(A) = 1$.

[31] Later on, you will see that 1) a default support of a discrete random variable is the set of values where the pmf is strictly positive and 2) a default support of a continuous random variable is the set of values where the pdf is strictly positive.

Example 7.20. Back to Example 7.7. Roll a fair dice. Set $\Omega = \{1, 2, 3, 4, 5, 6\}$. The dice is fair; therefore and the probability $P(\{\omega\}) = \frac{1}{6}$ for each outcome ω inside Ω.
Let $Y(\omega) = (\omega - 3)^2$.

ω	1	2	3	4	5	6
$Y(\omega)$	4	1	0	1	4	9

The possible values of Y are 0, 1, 4, 9

(a) Consider set $B_1 = \{0, 1, 4\}$. Is B_1 a support of Y?

B_1 is a support of Y if $P[Y \in B_1] = 1$

$P[Y \in B_1] = P(\{1,2,3,4,5\}) = \frac{5}{6} \neq 1$

So, set B_1 is not a support of Y.

(b) Consider set $B_2 = \{0, 1, 4, 9\}$. Is B_2 a support of Y?

First, find $P[Y \in B_2]$.

The outcomes ω that make $Y(\omega) \in \{0, 1, 4, 9\}$ are 1, 2, 3, ..., 6

$P[Y \in B_2] = 1$

So, set B_2 is a support of Y.

(c) Consider set $B_3 = \{0, 1, \underline{\underline{2}}, \underline{\underline{3}}, 4, 9\}$. Is B_3 a support of Y?

First, find $P[Y \in B_3]$.

The outcomes ω that make $Y(\omega) \in \{0, 1, 2, 3, 4, 9\}$ are 1, 2, 3, ..., 6

$P[Y \in B_3] = 1$

So, set B_3 is another support of Y.

Remarks:

(a) A random variable can have multiple supports.

- Note that the set $\mathbb{R}$ contains every real numbers, and hence it is always a support of any random variables.

(b) Some supports contain "uselesss" members.

We don't need "2" and "3" for B_3 to be a support of Y.

- Usually we want to get the "minimal" support.

Definition 7.21. The ***probability distribution*** is a description of the probabilities associated with the random variable.

Some advanced references define the **law** or **distribution** of the random variable X as a set function

$$P^X(B) \equiv P[X \in B]$$

which maps subsets of real numbers into their probability values. However, later on, we shall see that there are many functions that are also referred to as the "distribution" of X as well. They are all equivalent in the sense that they (almost surely) give the same information about probability concerning X.

7.22. At a certain point in most probability courses, the sample space is rarely mentioned anymore and we work directly with random variables. The sample space often "disappears" along with the "(ω)" of $X(\omega)$ but they are really there in the background.

7.23. There are three types of of random variables.

(a) The first type, which will be discussed in Section 8, is called ***discrete random variable***. In practice, to tell whether a random variable is discrete, one simple way is to consider the "possible" values of the random variable. If it can be limited to only a finite or countably infinite number of possibilities, then it is discrete.

(b) We will later discuss ***continuous random variables*** whose possible values can be anywhere in some intervals of real numbers.

(c) The last type of random variables is the hybrid (mixed) type.